

AN INTEGRAL APPROACH TO THE GARDNER-FISHER AND UNTWISTED DOWKER SUMS

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ABSTRACT. We present a new and elegant integral approach to computing the Gardner-Fisher trigonometric power sum, which is given by

$$S_{m,v} = \left(\frac{\pi}{2m}\right)^{2v} \sum_{k=1}^{m-1} \cos^{-2v} \left(\frac{k\pi}{2m}\right),$$

where m and v are positive integers. This method not only confirms the results obtained earlier by an empirical method, but it is also much more expedient from a computational point of view. By comparing the formulas from both methods, we derive several new interesting number theoretic results involving symmetric polynomials over the set of quadratic powers up to $(v-1)^2$ and the generalized cosecant numbers. The method is then extended to other related trigonometric power sums including the untwisted Dowker sum. By comparing both forms for this important sum, we derive new formulas for specific values of the Nörlund polynomials. Finally, by using the results appearing in the tables, we consider more advanced sums involving the product of powers of cotangent and tangent with powers of cosecant and secant respectively.

1. INTRODUCTION

In 1969 Gardner [9] stated that the finite sum of inverse powers of cosines

$$(1.1) \quad S_{m,v} = \left(\frac{\pi}{2m}\right)^{2v} \sum_{k=1}^{m-1} \cos^{-2v} \left(\frac{k\pi}{2m}\right),$$

where m and v are positive integers, emerges during the calculation of the v th cumulant of a certain quadratic form in m independent standardized normal variates. Although he was able to show that

$$\lim_{m \rightarrow \infty} S_{m,v} = \zeta(2v),$$

where ζ is the Riemann zeta function, he posed the problem of whether it was possible to obtain a “simpler” closed form expression for $S_{m,v}$, for all m and v .

In solving this problem, Fisher [6, 16] observed that the above sum could also be written as

$$(1.2) \quad S_{m,v} = \left(\frac{\pi}{2m}\right)^{2v} \sum_{k=1}^{m-1} \sin^{-2v} \left(\frac{k\pi}{2m}\right),$$

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and then devised an ingenious generating function approach for the more general trigonometric power series given by

$$Q_{m,v}(\delta) = \sum_{k=1}^{m-1} \sin^{-2v} \left(\frac{k\pi + \delta}{2m} \right).$$

Consequently, he found that

$$S_{m,1} = \frac{\pi^2}{6} \left(1 - \frac{1}{m^2} \right) \quad \text{and} \quad S_{m,2} = \frac{\pi^4}{90} \left(1 + \frac{5}{2m^2} - \frac{7}{2m^4} \right),$$

while for large m , he obtained

$$S_{m,v} = \zeta(2v) + \frac{v}{12m^2} \zeta(2v-2) + O(m^{-4}).$$

Although Gardner first obtained $S_{m,v}$, Fisher's solution is largely responsible for the continuing interest in the sum. Consequently, the sum will be henceforth referred to as the Gardner-Fisher sum.

2. GENERALIZED COSECANT NUMBERS

In a recent work one of us [18] showed that the Gardner-Fisher sum can be expressed as

$$S_{m,v} = \sum_{j=0}^{\infty} c_{2v,j} \left(\frac{\pi}{2m} \right)^{2j} \sum_{k=1}^{m-1} k^{2j-2v},$$

where the coefficients $c_{\rho,k}$ are a generalization of the cosecant numbers c_k in Ref. [19] and are given by

$$(2.1) \quad c_{\rho,k} = (-1)^k \sum_{\substack{n_1, n_2, n_3, \dots, n_k \\ \sum_{i=1}^k i n_i = k}}^{k, [k/2], [k/3], \dots, 1} (-1)^N (\rho)_N \prod_{i=1}^k \left(\frac{1}{(2i+1)!} \right)^{n_i} \frac{1}{n_i!}.$$

The above formula has been derived by applying the partition method for a power series expansion to $x^\rho / \sin^\rho x$. That is, the generalized cosecant numbers are the coefficients of the power series for this function. The partition method for a power series expansion was first introduced in Ref. [22], but more recently it has been developed further in Refs. [19]–[21] and [17]. To calculate the generalized cosecant number $c_{\rho,k}$ via (2.1), we need to determine the specific contribution made by each integer partition that sums to k . For example, if we wish to determine $c_{\rho,5}$ we require all the contributions made by the seven partitions that sum to 5 or those listed in the first column of Table 2.1. Depending on the function that is being studied, each element or part in a partition is assigned a specific value. In the case of $x^\rho / \sin^\rho x$, each element i is assigned a value of $(-1)^{i+1} / (2i+1)!$. Moreover, if an element occurs n_i times in a partition or possesses a multiplicity of n_i , then we need to take n_i values or $(-1)^{(i+1)n_i} / ((2i+1)!)^{n_i}$. Table 2.1 displays the multiplicities of all elements in the partitions that sum to 5.

Associated with each partition is a multinomial factor that is determined by taking the factorial of the total number of elements in the partition $N (= \sum_i n_i)$ and dividing by the factorials of all the multiplicities. For the partition $\{2, 1, 1, 1\}$ in Table 2.1, we have $n_1 = 3$ and $n_2 = 1$, while all other multiplicities do not contribute or vanish. Hence, the multinomial factor for this partition is $4! / (3! 1!) = 4$. When the function is accompanied by a power, another modification must be made. Each partition must be multiplied by Pochhammer factor of $\Gamma(N + \rho) / \Gamma(\rho) N!$ or $(\rho)_N / N!$. For $\rho = 1$, this simply yields unity

Partition	n_1	n_2	n_3	n_4	n_5
$\{5\}$					1
$\{4, 1\}$	1			1	
$\{3, 2\}$		1	1		
$\{3, 1, 1\}$	2		1		
$\{2, 2, 1\}$	1	2			
$\{2, 1, 1, 1\}$	3	1			
$\{1, 1, 1, 1, 1\}$	5				

TABLE 2.1. Multiplicities of the partitions whose elements or parts sum to 5

and thus, the multinomial factor is unaffected. Then the generalized cosecant numbers reduce to the cosecant numbers c_k [19], which are defined as

$$c_k = \frac{(-1)^{k+1}}{(2k)!} (2^{2k} - 2) B_{2k} = 2 (1 - 2^{1-2k}) \frac{\zeta(2k)}{\pi^{2k}},$$

and B_{2k} represent the Bernoulli numbers.

In (2.1) the product is concerned with the evaluating the contribution made by each partition based on the values of the multiplicities, while the sum is concerned with all partitions. Thus, the sum covers the range of values for the multiplicities. For example, n_1 attains a maximum value of k , which corresponds the partition with k ones, while n_2 attains a maximum value of $[k/2]$, which corresponds to the partition with $[k/2]$ twos in it. Here, $[k/n]$ denotes the floor function or the largest integer less than or equal to k/n . Therefore for odd values, the partition with $[k/2]$ twos would also have $n_1 = 1$. Hence, we see that n_i can only attain a maximum value of $[k/i]$, which becomes the upper limit for each multiplicity in (2.1). Furthermore, a valid partition must satisfy the constraint given by $\sum_{i=1}^k i n_i = k$.

As an example, let us consider the evaluation of $c_{\rho,5}$. According to Table 2.1 there are seven partitions whose contributions must be evaluated. By following the steps given above, we find that in the order in which the partitions appear in Table 2.1, (2.1) yields

$$(2.2) \quad \begin{aligned} c_{\rho,5} = & (\rho)_1 \frac{1}{11!} - \frac{(\rho)_2}{2!} \frac{2!}{1!1!} \frac{1}{3!9!} - \frac{(\rho)_2}{2!} \frac{2!}{1!1!} \frac{1}{5!7!} + \frac{(\rho)_3}{3!} \frac{3!}{1!2!} \frac{1}{3!27!} \\ & + \frac{(\rho)_3}{3!} \frac{3!}{1!2!} \frac{1}{5!23!} - \frac{(\rho)_4}{4!} \frac{4!}{1!3!} \frac{1}{3!35!} + \frac{(\rho)_5}{5!} \frac{5!}{5!} \frac{1}{3!^5}. \end{aligned}$$

From the above result we see that $c_{\rho,5}$ is a fifth order polynomial in ρ . In fact, the highest order term comes from the partition with k ones in it, which produces the term with $(\rho)_k$. Hence, we observe that the generalized cosecant numbers are polynomials of order k . Another interesting property is that all the contributions from partitions with the same number of elements in them possess the same sign, which toggles according to whether there is an even or odd number of elements in the partitions. Furthermore, (2.2) can be simplified further, whereupon one arrives at the result appearing in the sixth row below the headings in Table 2.2.

Table 2.2 displays the first fifteen generalized cosecant numbers obtained by determining the multiplicities of all the partitions that sum to each order k and then evaluating the sum of their contributions according to steps given above. Beyond $k = 10$, the partition method for a power series expansion becomes laborious due to the exponential increase in the number of partitions. To circumvent this problem, a general computing methodology is required to determine higher order coefficients via the partition method. This methodology, which is based on representing all the partitions that sum to a specific order as a tree diagram and invoking the bivariate recursive central partition (BRCP) algorithm,

k	$c_{\rho,k}$
0	1
1	$\frac{1}{3!} \rho$
2	$\frac{2}{6!} (2\rho + 5\rho^2)$
3	$\frac{8}{9!} (16\rho + 42\rho^2 + 35\rho^3)$
4	$\frac{2}{3 \cdot 10!} (144\rho + 404\rho^2 + 420\rho^3 + 175\rho^4)$
5	$\frac{4}{3 \cdot 12!} (768\rho + 2288\rho^2 + 2684\rho^3 + 1540\rho^4 + 385\rho^5)$
6	$\frac{2}{9 \cdot 15!} (1061376\rho + 3327594\rho^2 + 4252248\rho^3 + 2862860\rho^4 + 1051050\rho^5 + 175175\rho^6)$
7	$\frac{1}{27 \cdot 15!} (552960\rho + 1810176\rho^2 + 2471456\rho^3 + 1849848\rho^4 + 820820\rho^5 + 210210\rho^6 + 25025\rho^7)$
8	$\frac{2}{45 \cdot 18!} (200005632\rho + 679395072\rho^2 + 978649472\rho^3 + 792548432\rho^4 + 397517120\rho^5 + 125925800\rho^6 + 23823800\rho^7 + 2127125\rho^8)$
9	$\frac{4}{81 \cdot 21!} (129369047040\rho + 453757851648\rho^2 + 683526873856\rho^3 + 589153364352\rho^4 + 323159810064\rho^5 + 117327450240\rho^6 + 27973905960\rho^7 + 4073869800\rho^8 + 282907625\rho^9)$
10	$\frac{2}{6075 \cdot 22!} (38930128699392\rho + 140441050828800\rho^2 + 219792161825280\rho^3 + 199416835425280\rho^4 + 117302530691808\rho^5 + 47005085727600\rho^6 + 12995644662000\rho^7 + 2422012593000\rho^8 + 280078548750\rho^9 + 15559919375\rho^{10})$
11	$\frac{8}{243 \cdot 25!} (494848416153600\rho + 1830317979303936\rho^2 + 2961137042841600\rho^3 + 2805729689044480\rho^4 + 1747214980192000\rho^5 + 755817391389984\rho^6 + 232489541684400\rho^7 + 50749166067600\rho^8 + 7607466867000\rho^9 + 715756291250\rho^{10} + 32534376875\rho^{11})$
12	$\frac{2}{2835 \cdot 27!} (1505662706987827200\rho + 5695207005856038912\rho^2 + 9487372599204065280\rho^3 + 9332354263294766080\rho^4 + 6096633539052376320\rho^5 + 2806128331871953088\rho^6 + 937291839756592320\rho^7 + 229239926321406000\rho^8 + 40598842049766000\rho^9 + 5005999501002500\rho^{10} + 390802935022500\rho^{11} + 14803141478125\rho^{12})$
13	$\frac{232}{81 \cdot 30!} (844922884529848320\rho + 3261358271400247296\rho^2 + 5576528334428209152\rho^3 + 5668465199488266240\rho^4 + 3858582205451484160\rho^5 + 1870620248833400064\rho^6 + 667822651436228288\rho^7 + 178292330746770240\rho^8 + 35600276746834800\rho^9 + 5225593531158000\rho^{10} + 539680243602500\rho^{11} + 35527539547500\rho^{12} + 1138703190625\rho^{13})$
14	$\frac{2}{1215 \cdot 30!} (138319015041155727360\rho + 543855095595477762048\rho^2 + 952027796641042464768\rho^3 + 996352286992030556160\rho^4 + 703040965960031795200\rho^5 + 356312537387839432192\rho^6 + 134466795172062184832\rho^7 + 38526945410311117760\rho^8 + 8436987713444690400\rho^9 + 1404048942958662000\rho^{10} + 173777038440005000\rho^{11} + 15258232341852500\rho^{12} + 858582205731250\rho^{13} + 23587423234375\rho^{14})$
15	$\frac{1088}{729 \cdot 35!} (562009739464769840087040\rho + 2247511941596311764074496\rho^2 + 4019108379306905439830016\rho^3 + 4317745925208072594259968\rho^4 + 3145163776677939429416960\rho^5 + 1656917203539032341530624\rho^6 + 655643919364420586023424\rho^7 + 199227919419039256217472\rho^8 + 46995751664475880185920\rho^9 + 8614026107092938211680\rho^{10} + 1214778349162323946000\rho^{11} + 128587452922193265000\rho^{12} + 9720180867524627500\rho^{13} + 472946705787806250\rho^{14} + 11260635852090625\rho^{15})$

TABLE 2.2. Generalised cosecant numbers $c_{\rho,k}$ up to $k = 15$

is presented in Ref. [17]. The general expressions for the coefficients can then imported into a mathematical software package such as Mathematica [29] whereupon its symbolic routines yield the final values presented in Table 2.2.

For the special case, where ρ is an even integer (the case of interest here), the generalized cosecant numbers satisfy the following recurrence relation:

$$(2.3) \quad c_{2n+2,k+1} = \frac{(2k+2-2n)}{2n} \frac{(2k+1-2n)}{(2n+1)} c_{2n,k+1} + \frac{2n}{2n+1} c_{2n,k}.$$

This equation is obtained by introducing the power series expansion for $x^{2n}/\sin^{2n} x$ into (27) of Ref. [18], which is

$$(2.4) \quad \frac{d^2}{dx^2} \frac{1}{\sin^{2n} x} = \frac{2n}{\sin^{2n} x} + \frac{2n(2n+1) \cos^2 x}{\sin^{2n+2} x}.$$

Then one equates like powers of x . For the special case of $n = 1$, the numbers are related to the cosecant-squared numbers in Ref. [19]. Consequently, we find that

$$(2.5) \quad c_{2,k} = 2(2k-1) \frac{\zeta(2k)}{\pi^{2k}}.$$

Furthermore, the k -dependence in (2.3) can be removed by deducing that the $c_{2n,k}$ can be expressed generally as

$$(2.6) \quad c_{2n,k} = 4 \sum_{j=0}^{n-1} \frac{\Gamma(k-j)}{\Gamma(k-n+1)} \frac{\Gamma(k-j+1/2)}{\Gamma(k-n+1/2)} \frac{1}{\Gamma(n)} \frac{\Gamma(j+1/2)}{\Gamma(n+1/2)} C(n,j) \frac{\zeta(2k-2j)}{\pi^{2k-2j}},$$

where

$$(2.7) \quad C(n,j) = C(n-1,j) + \frac{(n-1)^2}{j-1/2} C(n-1,j-1),$$

and $C(1,0) = 2$ from equating (2.6) with (2.5). For $n = 1$, we find that $C(n,1) = 2 \sum_{l=1}^{n-1} l^2 = 4B_3(n)/3$, where $B_k(x)$ denotes a Bernoulli polynomial. Introducing this result into (2.7) yields $C(n,2)$, which is given by

$$(2.8) \quad C(n,2) = \frac{1}{135} n(n-1)(n-2)(2n-1)(2n-3)(5n+1) = \frac{1}{6} (2n-4)_4 c_{2n,2}.$$

Similarly, if one introduces (2.8) into (2.7), then one obtains

$$(2.9) \quad C(n,3) = \frac{1}{60} (2n-6)_6 c_{2n,3}.$$

Therefore, we see that the $C(n,j)$ are related to the generalized cosecant numbers. In the next section we shall also see that the generalized cosecant numbers are related to the symmetric polynomials over the set of positive square integers.

3. INTEGRAL APPROACH

In this section we present a totally different approach to evaluating the Gardner-Fisher sum from others who have studied similar trigonometric power sums, [1]- [8], [10]- [12], [14], [16] and [24]- [25]. The main advantage of the integral approach presented here is that it yields a final form for the sum that is more expedient from a computational point of view than the other references. In addition, except for Ref. [18], it is both more informative and compact than the other references. It also provides an independent corroboration of the results in Ref. [18], which were obtained by empirical means. As a consequence, a comparison of the results can be undertaken.

We begin this section by differentiating No. 8.365(10) in Ref. [13], which yields

$$(3.1) \quad \pi^2 \csc^2(\pi x) = \psi'(x) + \psi'(1-x).$$

By introducing the series form for the derivative of the digamma function, viz. $\psi'(x) = \sum_{n=0}^{\infty} 1/(n+x)^2$ from No. 8.363(8) of Ref. [13], we replace the summand in (3.1) by the integral representation for the gamma function. On interchanging the order of the sum and the integral, we obtain

$$\pi^2 \csc^2(\pi x) = \int_0^{\infty} \sum_{n=0}^{\infty} (e^{-(n+x)t} + e^{-(n+1-x)t}) t dt .$$

Next we evaluate the sums over n via the geometric series. This yields

$$\pi^2 \csc^2(\pi x) = \int_0^{\infty} \left(\frac{e^{-xt}}{1-e^{-t}} + \frac{e^{-(1-x)t}}{1-e^{-t}} \right) t dt .$$

Now we make the change of variable $u = e^{-t}$, thereby arriving at

$$(3.2) \quad \pi^2 \csc^2(\pi x) = - \int_0^1 \frac{\ln u}{1-u} (u^x + u^{1-x}) \frac{du}{u} .$$

With the aid of the following identity:

$$\prod_{n=1}^{v-1} (\partial_z^2 + 4n^2) \csc^2 z = (2v-1)! \csc^{2v} z, \quad \text{for } v = 1, 2, 3, \dots,$$

which is obtained by replacing the $\cos^2 x$ by $1 - \sin^2 x$ in (2.4), thereby yielding

$$\left(\frac{d^2}{dx^2} + 4n^2 \right) \csc^{2n} x = 2n(2n+1) \csc^{2n+2} x ,$$

and then multiplying this resulting equation by successive values of n , we find after an elementary change of variable that (3.2) can be expressed as

$$\csc^{2v} \left(\frac{k\pi}{2m} \right) = - \frac{4^v m^2}{(2v-1)! \pi^2} \int_0^1 \frac{\ln u}{1-u^{2m}} \prod_{n=1}^{v-1} \left(\frac{m^2}{\pi^2} \ln^2 u + n^2 \right) (u^k + u^{2m-k}) \frac{du}{u} .$$

Carrying out the summation over k in the Gardner-Fisher sum or (1.2) yields

$$(3.3) \quad \sum_{k=1}^{m-1} \csc^{2v} \left(\frac{k\pi}{2m} \right) = - \frac{4^v m^2}{(2v-1)! \pi^2} \int_0^1 \prod_{n=1}^{v-1} \left(\frac{m^2}{\pi^2} \ln^2 u + n^2 \right) \frac{(1-u^{m-1})}{(1-u^m)(1-u)} du .$$

If we introduce Newton's identities for symmetric polynomials [27], then (3.3) becomes

$$(3.4) \quad S_{m,v} = - \frac{4^v}{(2v-1)!} \sum_{n=0}^{v-1} s(v, n) \left(\frac{\pi}{m} \right)^{2n-2v} \int_0^1 \frac{(1-u^{m-1})}{(1-u^m)(1-u)} \ln^{2v-2n-1} u du ,$$

where $s(v, n)$ represents the n th elementary symmetric polynomial obtained by summing quadratic powers, viz. $1^2, 2^2, \dots, (v-1)^2$. That is,

$$s(v, n) = \sum_{1 \leq i_1 < i_2 < \dots < i_n < v-1} x_{i_1} x_{i_2} \dots x_{i_n} ,$$

where $x_{i_1} < x_{i_2} < \dots < x_{i_n}$ and each x_{i_j} is equal to at least one value in the set $\{1, 2^2, 3^2, \dots, (v-1)^2\}$. In particular, for the three lowest values of n , they are given by

$$s(v, 0) = 1, \quad s(v, 1) = (v-1)v(2v-1)/6 ,$$

and

$$s(v, 2) = \frac{(5v+1)}{4 \cdot 6!} (2v-4)_5 ,$$

while for the three highest values of n , they are given by

$$s(v, v-1) = (v-1)!^2, \quad s(v, v-2) = (v-1)!^2 (\zeta(2) - \zeta(2, v)),$$

and

$$s(v, v-3) = \frac{(v-1)!^2}{2} \left((\zeta(2) - \zeta(2, v))^2 + \zeta(4, v) - \zeta(4) \right).$$

The integral I over u can now be evaluated by decomposing the denominator of the integrand and applying No. 4.271(4) from Ref. [13]. Then we find that

$$I = \Gamma(2v-2n) \zeta(2v-2n) (m^{-2(v-n)} - 1).$$

Introducing this result into (3.4), we finally arrive at

$$(3.5) \quad S_{m,v} = \frac{1}{(2v-1)!} \sum_{n=0}^{v-1} \left(\frac{\pi}{m} \right)^{2n} s(v, n) \Gamma(2v-2n) \zeta(2v-2n) \left(1 - \frac{1}{m^{2v-2n}} \right),$$

The above result for the Gardner-Fisher sum is computationally expedient because it can be implemented as a one line instruction in Mathematica [29]. With the aid of the SymmetricPolynomial and the gamma and zeta function routines, (3.5) can be expressed in Mathematica as

$$\begin{aligned} S[m_-, v_-] &:= (1/((2 v - 1)!)) \text{Sum}[(\text{Pi}/m)^(2 n) \\ &\text{SymmetricPolynomial}[n, \text{Table}[k^2, k, 1, v - 1]] \\ &\text{Gamma}[2 v - 2 n] \text{Zeta}[2 v - 2 n] (1 - 1/(m^(2 v - 2 n))), \{n, 0, v - 1\}] \end{aligned}$$

If we type in the instruction

$$F[m, n] := \text{Table}[\text{Simplify}[S[m, v]], \{v, 1, n\}]$$

then we can construct a table of values of the Gardner-Fisher sum. In fact, by typing

$$\text{Timing}[\text{Grid}[\text{Partition}[F[m, 15], 1]]]$$

one obtains the first 15 values of the Gardner-Fisher sum printed out in table form. For a Venom Blackbook 17S Pro High Performance laptop with 8 Gb RAM and equipped with Mathematica 10.1 these take less than 0.16 seconds to appear on the screen. Table 3.1 presents the output generated by the above instruction. We have not only been able to present a much greater number of results than Table 2 of Ref. [18], but we have also corrected the typographical error occurring in the last term of the second result, which should be $-7/2m^4$, not $-7/2m^2$. The results have been presented in terms of the zeta function, thereby maintaining consistency with Gardner's limit mentioned in the introduction. Moreover, the results can be expressed as

$$S_{m,v} = (2m^2)^{-v} (m^2 - 1) p_{v-1}(m^2),$$

where $p_v(x)$ is a polynomial of degree v with the coefficients $p_{v,j}$ satisfying

$$(3.6) \quad \begin{aligned} p_{v,v} &= 2^{v+1} \zeta(2v+2); \quad p_{v,v-1} = 2^{v+1} \zeta(2v+2) + 2^{v-1} \pi^2 \zeta(2v)/3, \\ p_{v,j-1} &= p_{v,j} + \frac{2^{v+1}}{(2v+1)!} \pi^{2v+2-2j} s(v+1, v+1-j) \Gamma(2j) \zeta(2j), \end{aligned}$$

and

$$(3.7) \quad p_{v,0} = \frac{2^{v+1}}{(2v+1)!} \sum_{j=0}^v \pi^{2j} s(v+1, j) \Gamma(2v+2-2j) \zeta(2v+2-2j).$$

v	$(2m^2)^v S_{m,v}/(m^2 - 1)$
1	$2\zeta(2)$
2	$2\zeta(4)(7 + 2m^2)$
3	$\zeta(6)(71 + 29m^2 + 8m^4)$
4	$\frac{2}{3}\zeta(8)(521 + 251m^2 + 104m^4 + 24m^6)$
5	$\zeta(10)(1693 + 901m^2 + 450m^4 + 164m^6 + 32m^8)$
6	$\frac{1}{691}\zeta(12)(5710469 + 3253469m^2 + 1815032m^4 + 821182m^6 + 262624m^8 + 44224m^{10})$
7	$\frac{1}{30}\zeta(14)(1212457 + 726457m^2 + 436531m^4 + 224891m^6 + 91472m^8 + 25952m^{10} + 3840m^{12})$
8	$\frac{2}{10851}\zeta(16)(1074010337 + 669172337m^2 + 424359179m^4 + 238674979m^6 + 112425856m^8 + 41352256m^{10} + 10528128m^{12} + 1388928m^{14})$
9	$\frac{1}{219335}\zeta(18)(212920335247 + 136906047247m^2 + 90463687339m^4 + 54230674609m^6 + 28237533526m^8 + 12198655216m^{10} + 4088714368m^{12} + 943572928m^{14} + 112299520m^{16})$
10	$\frac{1}{3666831}\zeta(20)(17471801743019 + 11530685023019m^2 + 7871509097999m^4 + 4954255382249m^6 + 2775847146014m^8 + 1339874175764m^{10} + 533660885024m^{12} + 164177309024m^{14} + 34637202944m^{16} + 3754834944m^{18})$
11	$\frac{22}{51270780}\zeta(22)(54543344015461 + 36797151215461m^2 + 25796160434461m^4 + 16874095108441m^6 + 10001178676906m^8 + 5238677224486m^{10} + 2352811821856m^{12} + 868592118736m^{14} + 246835895296m^{16} + 47953445376m^{18} + 4772843520m^{20})$
12	$\frac{1}{3545461365}\zeta(24)(407813841938063843 + 280376888294063843m^2 + 200956537702208843m^4 + 135642820502091743m^6 + 84063888200850398m^8 + 46897459147087298m^{10} + 23026002351214568m^{12} + 9661082872248968m^{14} + 3322054926238208m^{16} + 876937263364608m^{18} + 157850700656640m^{20} + 14522209751040m^{22})$
13	$\frac{1}{82899306}\zeta(26)(46885126608855949 + 32769033282135949m^2 + 23932490769301549m^4 + 16581373742311501m^6 + 10657001539527466m^8 + 6251551315383406m^{10} + 3288926574739816m^{12} + 1518316248517036m^{14} + 597409832798656m^{16} + 192183289614336m^{18} + 47355550989312m^{20} + 7940215990272m^{22} + 679111114752m^{24})$
14	$\frac{1}{6785560294}\zeta(28)(18876870638218467389 + 13386158464071267389m^2 + 9936028183278303389m^4 + 7037937839930870909m^6 + 4662484330428335720m^8 + 2849737068204025620m^{10} + 1584446206207320450m^{12} + 787977205890303150m^{14} + 343067669320476480m^{16} + 127034728547601280m^{18} + 38382891063494656m^{20} + 8866606540512256m^{22} + 1391299071164416m^{24} + 111174619856896m^{26})$
15	$\frac{1}{6892673020804}\zeta(30)(94347422281763617652599 + 67773208432462419572599m^2 + 51020868251978707090999m^4 + 36832275607229172708727m^6 + 25034877087339771576013m^8 + 15834856690259096336845m^{10} + 9212522702209442756325m^{12} + 4863979571266245490245m^{14} + 2292234370375168188000m^{16} + 943898600252928069440m^{18} + 329986597479890425856m^{20} + 93975868433656203776m^{22} + 20429529466374004736m^{24} + 3012284194329018368m^{26} + 225859109545705472m^{28})$

TABLE 3.1. The first 15 $(2m^2)^v S_{m,v}/(m^2 - 1)$ for v , a positive integer

As mentioned previously, values of the Gardner-Fisher sum have been obtained in Ref. [18] by an empirical approach. According to this approach, the coefficients of m^{-2i} in $S_{m,v}$ for $i < v$ were determined to be

$$(3.8) \quad C_i^v = c_{2v,i} \zeta(2v - 2i) \left(\frac{\pi}{2}\right)^{2i},$$

while for $i = v$, they were given by

$$(3.9) \quad C_v^v = 2^v \left(\frac{\pi}{2}\right)^{2v} - \sum_{i=0}^{v-1} 2^{2v-2i} c_{2v,i} \left(\frac{\pi}{2}\right)^{2i} \zeta(2v-2i).$$

In these results we have dropped dividing by $\zeta(2v)$, which occurs when this factor is taken outside the results for $S_{m,v}$ to confirm Gardner's limit. In addition, (3.9) has an extra factor of 2^v in the first term on the rhs, which is missing in (44) of Ref. [18]. Moreover, in a future work it will be shown that the final coefficient of $S_{m,v}$ can also be expressed as

$$(3.10) \quad C_v^v = -\frac{1}{2}(c_{2v,v} + 1) \left(\frac{\pi}{2}\right)^{2v}.$$

By equating like powers of m^2 between the above results and (3.5) we find that

$$(3.11) \quad c_{2v,i} = 2^{2i} \frac{\Gamma(2v-2i)}{\Gamma(2v)} s(v,i), \quad i < v,$$

$$(3.12) \quad \sum_{i=0}^{v-1} \left(2^{2v-2i} c_{2v,i} - \frac{2^{2i}}{(2v-1)!} s(v,i) \Gamma(2v-2i) \right) \left(\frac{\pi}{2}\right)^{2i-2v} \zeta(2v-2i) = 2^v,$$

and

$$(3.13) \quad \sum_{n=0}^{v-1} \pi^{2n} s(v,n) \Gamma(2v-2n) \zeta(2v-2n) = \frac{1}{2} \Gamma(2v) (c_{2v,v} + 1) \left(\frac{\pi}{2}\right)^{2v}.$$

These interesting results involving the generalized cosecant numbers have been verified by programming them in Mathematica. Furthermore, by introducing (3.11) into (2.3), we obtain the recurrence relation for the symmetric polynomials, which is

$$s(n+1, k+1) = s(n, k+1) + n^2 s(n, k).$$

4. UNTWISTED DOWKER SUM

The integral approach of the previous section can be extended to the situation where the external normalization factor in the Gardner-Fisher sum is dropped and $\pi/2$ inside the trigonometric power is replaced by π/ℓ , where ℓ is any integer except 0. Then by using the same integral approach as before, we obtain

$$\sum_{k=1}^{m-1} \csc^{2v} \left(\frac{k\pi}{\ell} \right) = \frac{1}{\pi^{2v} \Gamma(2v)} \sum_{n=1}^{v-1} (2\pi)^{2n} s(v,n) \sum_{k=1}^{m-1} \psi^{2v-2n-1} \left(\frac{2k}{\ell m} + \frac{\ell-1}{\ell} \right).$$

With the aid of No. 8.363(8) in Ref. [13], we can replace the derivative of digamma function by the Hurwitz zeta function, thereby arriving at

$$(4.1) \quad S_{m,v,\ell} = \sum_{k=1}^{m-1} \csc^{2v} \left(\frac{k\pi}{\ell} \right) = \frac{1}{\pi^{2v} \Gamma(2v)} \sum_{n=0}^{v-1} (2\pi)^{2n} s(v,n) \sum_{k=1}^{m-1} \Gamma(2v-2n) \times \zeta^{2v-2n} \left(2v-2n, 1 + (2k/m-1)/|\ell| \right).$$

For $|\ell| \neq 1$ or 2 , the Hurwitz zeta function is intractable and consequently, this generalization of the Gardner-Fisher sum will not yield polynomials as in Table 3.1. However,

for the important case of $\ell = 1$, which is studied extensively in Ref. [4] and is known as the untwisted form of the Dowker sum [5], (4.1) reduces to

$$(4.2) \quad S_{m,v,1} = \sum_{k=1}^{m-1} \csc^{2v} \left(\frac{k\pi}{m} \right) = 2^{2v+1} \sum_{n=0}^{v-1} \left(\frac{m}{2\pi} \right)^{2v-2n} \frac{\Gamma(2v-2n)}{\Gamma(2v)} \\ \times s(v, n) \left(1 - \frac{1}{m^{2v-2n}} \right) \zeta(2v-2n).$$

Because there is no normalization factor outside the sum as in the Gardner-Fisher sum, we obtain polynomials in powers of m^2 of degree v . Although these polynomials are denoted by $C_{2v}(m)$ in Ref. [4], we shall denote them by $q_v(m^2)$ with the coefficients of m^{2i} represented by $q_{v,i}$. From (4.2) we find that

$$(4.3) \quad q_{v,0} = -2^{2v+1} \sum_{n=0}^{v-1} (2\pi)^{2n-2v} \frac{\Gamma(2v-2n)}{\Gamma(2v)} s(v, n) \zeta(2v-2n), \quad q_{v,1} = \frac{1}{6} \frac{\Gamma(v) \Gamma(1/2)}{\Gamma(v+1/2)},$$

$$(4.4) \quad q_{v,i} = \frac{2^{2v-2i+1}}{\pi^{2i}} \frac{\Gamma(2i)}{\Gamma(2v)} s(v, v-i) \zeta(2i), \quad i < v, \quad \text{and} \quad q_{v,v} = \frac{2}{\pi^{2v}} \zeta(2v).$$

Moreover, we can use (3.11) to express $q_{v,0}$ and $q_{v,i}$ in terms of the generalized cosecant numbers. Hence, we obtain

$$q_{v,0} = -2 \sum_{n=0}^{v-1} \pi^{2n-2v} c_{2v,n} \zeta(2v-2n),$$

and

$$q_{v,i} = \frac{2}{\pi^{2i}} c_{2v,v-i} \zeta(2i), \quad i < v.$$

As in the case of the Gardner-Fisher sum, the polynomials obtained from (4.2) possess a common factor of $(m^2 - 1)$. Consequently, we can simplify the presentation of the polynomials for $S_{m,v,1}$ by removing this factor. Hence, Table 4.1 presents the first 15 values of $S_{m,v,1}/(m^2 - 1)$, which were obtained by writing (4.2) as a one-line instruction in Mathematica as we did for the Gardner-Fisher sum. In this instance the instruction becomes

```
CS[m_, v_] := (2^(2 v + 1)/(2 v - 1)!) Sum[(2 Pi/m)^(2 n - 2 v)
SymmetricPolynomial[n, Table[k^2, {k, 1, v - 1}]]
Gamma[2 (v - n)] Zeta[2 (v - n)] (1 - 1/(m^(2 (v - n))))], {n, 0, v - 1}]
```

Then we can use the same intructions below the instruction for the Gardner-Fisher sum except that $S[m,v]$ is now replaced by $CS[m,v]$ to tabulate the polynomials and time the calculation. The results in Table 4.1 took 0.12 seconds to compute as the same Venom laptop mentioned previously. It should also be mentioned that the first five results in the table are identical to those given in Ref. [4]. In fact, these authors prove that

$$(4.5) \quad q_v(m^2) = (-1)^{v-1} \frac{2^{2v}}{(2v)!} \sum_{n=0}^v \binom{2v}{2n} B_{2v-2n} B_{2n}^{(2v)}(v) m^{2v-2n},$$

where B_{2v-2n} are the ordinary Bernoulli numbers and $B_k^{(m)}(x)$ are the Bernoulli polynomials of order m and degree k and sometimes referred as to Nörlund polynomials. By

v	$S_{m,v,1}/(m^2 - 1)$
1	$\frac{2\zeta(2)}{\pi^2}$
2	$\frac{2\zeta(4)}{\pi^4}(m^2 + 11)$
3	$\frac{\zeta(6)}{\pi^6}(2m^4 + 23m^2 + 191)$
4	$\frac{2\zeta(8)}{3\pi^8}(m^2 + 11)(3m^4 + 10m^2 + 227)$
5	$\frac{\zeta(10)}{\pi^{10}}(2m^8 + 35m^6 + 321m^4 + 2125m^2 + 14797)$
6	$\frac{2\zeta(12)}{1382\pi^{12}}(1382m^{10} + 28682m^8 + 307961m^6 + 2295661m^4 + 13803157m^2 + 92427157)$
7	$\frac{2\zeta(14)}{60\pi^{14}}(60m^{12} + 1442m^{10} + 17822m^8 + 151241m^6 + 997801m^4 + 5636617m^2 + 36740617)$
8	$\frac{2\zeta(16)}{10851\pi^{16}}(10851m^{14} + 296451m^{12} + 4149467m^{10} + 39686267m^8 + 292184513m^6 + 1777658113m^4 + 9611679169m^2 + 61430943169)$
9	$\frac{2\zeta(18)}{438670\pi^{18}}(438670m^{16} + 13427317m^{14} + 209998657m^{12} + 2237483869m^{10} + 18276362179m^8 + 122248926511m^6 + 701977130191m^4 + 3674288164303m^2 + 23133945892303)$
10	$\frac{2\zeta(20)}{7333662\pi^{20}}(7333662m^{18} + 248602162m^{16} + 4296730477m^{14} + 50482177477m^{12} + 453588822847m^{10} + 3325534763347m^8 + 20752800653227m^6 + 114104919557227m^4 + 582479437959787m^2 + 3624331198599787)$
11	$\frac{2\zeta(22)}{4660980\pi^{22}}(4660980m^{20} + 173335206m^{18} + 3280873486m^{16} + 42140637451m^{14} + 413195563231m^{12} + 3299060965861m^{10} + 22349066775541m^8 + 132315729680101m^6 + 703327910545381m^4 + 3519581550481381m^2 + 21691682977681381)$
12	$\frac{2\zeta(24)}{7090922730\pi^{24}}(7090922730m^{22} + 287029381530m^{20} + 5904893152686m^{18} + 82314820117486m^{16} + 874693313368831m^{14} + 7557153052851631m^{12} + 55300066644597091m^{10} + 352631499074701891m^8 + 2003157332714424931m^6 + 10363313134329413731m^4 + 51026532637359173731m^2 + 312017413700271173731)$
13	$\frac{2\zeta(26)}{165798612\pi^{26}}(165798612m^{24} + 7256721342m^{22} + 161222873682m^{20} + 2424156289698m^{18} + 27750815238718m^{16} + 257977419168313m^{14} + 2028587745391093m^{12} + 13879086707965453m^{10} + 84366290294270413m^8 + 463526111272448653m^6 + 2345412070181900941m^4 + 11394031603324326541m^2 + 69213549869569446541)$
14	$\frac{2\zeta(28)}{13571120588\pi^{28}}(13571120588m^{26} + 638631887828m^{24} + 15238841788898m^{22} + 245834814624698m^{20} + 3016204736003030m^{18} + 30020322332612430m^{16} + 252475090617525765m^{14} + 1845413091251560365m^{12} + 11967739987225201725m^{10} + 69975652378403124925m^8 + 374033701594727629117m^6 + 1857855957388613058877m^4 + 8923722772452603330877m^2 + 53903636903066465730877)$
15	$\frac{2\zeta(30)}{13785346041608\pi^{30}}(13785346041608m^{28} + 694064907756284m^{26} + 17703093493737716m^{24} + 304993480084683806m^{22} + 3992661121432093526m^{20} + 42362161294746946250m^{18} + 379446103825306975890m^{16} + 2951191304716384278135m^{14} + 20345363828489173342455m^{12} + 126302707637283630630775m^{10} + 715104013050446845937527m^8 + 3735238034142133535912311m^6 + 18264356902365656743358839m^4 + 86881942281626943067992439m^2 + 522273861988577772410712439)$

TABLE 4.1. The first 15 $S_{m,v,1}/(m^2 - 1)$ for v , a positive integer

equating like powers of m between (4.2) and (4.5), we obtain the following results:

$$(4.6) \quad B_{2n}^{(2v)}(v) = (-1)^n (2n)! \frac{\Gamma(2v - 2n)}{\Gamma(2v)} s(v, n), \quad n < v,$$

and

$$(4.7) \quad B_{2v}^{(2v)}(v) = (-1)^v 4v \sum_{n=0}^{v-1} (2\pi)^{2n-2v} \Gamma(2v-2n) s(v, n) \zeta(2v-2n) .$$

In obtaining this result we have used No. 9.616 in Ref. [13], which expresses the Bernoulli numbers in terms of the Riemann zeta function. Furthermore, introducing (3.11) into (4.6) and (4.7) yields

$$B_{2n}^{(2v)}(v) = (-1)^n 2^{-2n} (2n)! c_{2v,n} , \quad n < v ,$$

and

$$B_{2v}^{(2v)}(v) = (-1)^v 2^{1-2v} \Gamma(2v+1) \sum_{n=0}^{v-1} \pi^{2n-2v} c_{2v,n} \zeta(2v-2n) .$$

The above results can be verified in Mathematica [29], where the Bernoulli polynomials of order m and degree k are determined by using the NorlundB routine.

5. OTHER SUMS

We can use the results of the previous sections to consider more intricate trigonometric power sums than the Gardner-Fisher and untwisted Dowker sums. E.g., consider the following series:

$$(5.1) \quad S_{m,v,w,\ell}^{CC} := \sum_{k=1}^{m-1} \cot^{2v} \left(\frac{k\pi}{\ell m} \right) \csc^{2w} \left(\frac{k\pi}{\ell m} \right) ,$$

where $v \geq 0$ and $w \geq 0$ excluding $v + w = 0$, while $\ell = 1$ and $\ell = 2$ correspond respectively to the untwisted Dowker and Gardner-Fisher cases. We could introduce a factor of $\cos(2ak\pi/\ell m)$, where a is an integer less than $\ell m - 1$, into the summand. For $\ell = 1$, this becomes an extension of the twisted Dowker sum [5], which we aim to study in a future work.

The above trigonometric power sum can also be expressed as

$$(5.2) \quad S_{m,v,w,\ell}^{CC} = \sum_{k=1}^{m-1} \frac{\cos^{2v}(k\pi/\ell m)}{\sin^{2v+2w}(k\pi/\ell m)} .$$

Now we replace the cosine power in the numerator by $(1 - \sin^2(k\pi/\ell m))^v$ and apply the binomial theorem [28], thereby obtaining

$$(5.3) \quad S_{m,v,w,\ell}^{CC} = \sum_{j=0}^v \sum_{k=1}^{m-1} (-1)^{v-j} \binom{v}{j} \csc^{2w+2j} \left(\frac{k\pi}{\ell m} \right) .$$

Therefore, the sum represents a finite sum of Fisher-Gardner and untwisted Dowker sums depending upon the value of ℓ .

If $w + v \leq 15$, then we can use the results displayed in Tables 3.1 and 4.1 to obtain the values of $S_{m,v,w,\ell}^{CC}$. Moreover, denoting the quantities listed in Table 3.1 as $R_v(m^2)$, i.e. $R_v(m^2) = (2m^2)^v S_{m,v}/(m^2 - 1)$, we find that $S_{m,v,w,2}^{CC}$ reduces to

$$(5.4) \quad S_{m,v,w,2}^{CC} = (m^2 - 1) \sum_{j=0}^v (-1)^{v-j} \binom{v}{j} \frac{2^{w+j}}{\pi^{2w+2j}} R_{w+j}(m^2) .$$

If v and w are set equal to 5 and 4 respectively, then using the values of $R_4(m^2)$ to $R_9(m^2)$ in Table 3.1 and the above result, we find that

$$(5.5) \quad \begin{aligned} S_{m,5,4,2}^{CC} = & \frac{16}{194896477400625} (m^2 - 1) (4m^2 - 1) (2280413161 + 712556555m^2 \\ & - 2906805048m^4 - 2535353600m^6 + 2920623488m^8 + 2565749760m^{10} \\ & - 3310462976m^{12} + 898396160m^{14}) . \end{aligned}$$

A general formula for the $\ell = 2$ case can be obtained by using Eqs. (3.8) to (3.10), which represent the coefficients of $S_{m,v}$. Then we arrive at

$$(5.6) \quad S_{m,v,w,2}^{CC} = \sum_{j=0}^v (-1)^{v-j} \binom{v}{j} \frac{2^{2v+2j}}{\pi^{2v+2j}} \sum_{i=0}^{v+j} C_i^{v+j} m^{2v+2j-2i} .$$

The above result can also be expressed in terms of the symmetric polynomials of quadratic integers since from (3.8), (3.10), (3.11) and (3.13), we have

$$C_i^v = (2v)_{-2n} \zeta(2v - 2i) s(v, i) , \quad i < v ,$$

and

$$C_v^v = - \sum_{n=0}^{v-1} \pi^{2n} (2v)_{-2n} \zeta(2v - 2n) s(v, n) .$$

For $\ell = 1$ or the untwisted Dowker case, we can use the results of Table 4.1 to obtain the values of the sums, $S_{m,v,w,1}^{CC}$, when $v + w \leq 15$. If we denote the quantities listed in the table as $T_v(m^2)$, i.e. $T_v(m^2) = S_{m,v,1}/(m^2 - 1)$, then we find that

$$(5.7) \quad S_{m,v,w,1}^{CC} = (m^2 - 1) \sum_{j=0}^v (-1)^{v-j} \binom{v}{j} T_{w+j}(m^2) .$$

For $v = 6$ and $w = 3$, the above result with the aid of the respective values in Table 4.1 yields

$$(5.8) \quad \begin{aligned} \sum_{k=1}^{m-1} \cot^{12}\left(\frac{k\pi}{m}\right) \csc^6\left(\frac{k\pi}{m}\right) = & \frac{1}{194896477400625} (m^2 - 1) (m^2 - 4) (-29787342748 \\ & + 1960688815m^2 + 3595494399m^4 - 275848135m^6 - 335395979m^8 \\ & + 98107275m^{10} - 10795297m^{12} + 438670m^{14}) . \end{aligned}$$

Both (5.5) and (5.8) have been checked for specific values of m by calculating the decimal values of the sums on the lhs and comparing them with the decimal values of the rational quantities on the rhs's in Mathematica. In terms of the $q_{v,i}$ given by (4.3) and (4.4), we can express $S_{m,v,w,1}^{CC}$ more generally as

$$(5.9) \quad S_{m,v,w,1}^{CC} = \sum_{j=0}^v (-1)^{v-j} \binom{v}{j} \sum_{i=0}^{w+j} q_{w+j,i} m^{2i} .$$

Another trigonometric power sum that can be studied with the aid of the previous sections is:

$$(5.10) \quad S_{m,v,w,\ell}^{TS} := \sum_{k=1}^{m-1} \tan^{2v}\left(\frac{k\pi}{\ell m}\right) \sec^{2w}\left(\frac{k\pi}{\ell m}\right) ,$$

where the same conditions apply on v and w as in (5.1). For $\ell = 2$, if we replace k by $m - k$, then we find that $S_{m,v,w,2}^{TS} = S_{m,v,w,2}^{CC}$. Hence, there is no need to consider this case.

On the other hand, for $\ell = 1$, there is a possibility that one of the values of k can produce powers of $\cos(\pi/2)$ in the denominator when m is an even integer. Therefore, this value of k needs to be excluded when $m = 2n$, where n is a non-zero integer. Therefore, we modify the above sum to

$$(5.11) \quad S_{2n,v,w,1}^{TS} = \sum_{\substack{k=1 \\ k \neq n}}^{2n-1} \tan^{2v} \left(\frac{k\pi}{2n} \right) \sec^{2w} \left(\frac{k\pi}{2n} \right) ,$$

We now split the above sum into two separate sums, the first ranging from $k = 1$ to $n - 1$ and the second, from $k = n - 1$ to $2n$. In the second sum we replace k by $2n - k$, which yields the first sum again. Consequently, (5.11) reduces to

$$(5.12) \quad S_{2n,v,w,1}^{TS} = 2 \sum_{k=1}^{n-1} \tan^{2v} \left(\frac{k\pi}{2n} \right) \sec^{2w} \left(\frac{k\pi}{2n} \right) ,$$

By replacing k by $n-k$, we find that $S_{2n,v,w,1}^{TS} = 2S_{n,v,w,2}^{CC}$. The case of $m = 2n+1$ can also be reduced, but it yields a second trigonometric power sum with an alternating summand. Such sums will be studied in a future work. Finally, we add that the $\ell = 1$, $w = 0$ and $m = 2n+1$ case of (5.10) has been studied by Shevelev and Moses in Ref. [26], where they give the polynomial values of the sum for the first five values of v .

6. CONCLUSION

In this paper we have presented a new integral approach for evaluating the Gardner-Fisher sum or $S_{m,v}$ as defined by either (1.1) or (1.2), which is not only computationally expedient compared with other methods, but has also enabled us to quantify the coefficients of the polynomials as evidenced by (3.6) to (3.7). In so doing, we have been able to relate the generalized cosecant numbers of Ref. [19] to the symmetric polynomials $s(v, n)$ over the set of quadratic powers, $\{1, 2^2, 3^2, \dots, (v-1)^2\}$, via (3.11), which was achieved by matching the general result given here by (3.5) with the empirically determined results of (43) and (44) in Ref. [18].

To demonstrate the versatility of our integral approach, it was then extended to situations where $\pi/2$ in the trigonometric power of the Gardner-Fisher sum was replaced by π/ℓ . As a consequence, we were able to evaluate the sums for the $\ell = 1$ case or $S_{m,v,1}$, which is known as the untwisted Dowker sum [5] and has been studied extensively by Cvijović and Srivastava [4]. The latter authors obtain a general result for the sum, which is given by another unwieldy sum whose summand is a product of Bernoulli numbers and the esoteric Nörlund polynomials as in (4.5). As a result, one is unable to ascertain the mathematical forms for the coefficients of the polynomials in $S_{m,v,1}$. Hence, one has to rely on a software package such as Mathematica to generate the final forms for the untwisted Dowker sum. Nevertheless, the five values presented in [4] agree with the first five results in Table 4.1. Furthermore, by comparing their form for the untwisted Dowker sum with our (4.2), we are able to express the particular values of the Nörlund polynomials in their result either in terms of the specific symmetric polynomials $s(v, n)$ presented here or in terms of the generalized cosecant numbers. By using the results of the previous sections we were able to sketch out the calculations for more intricate sums involving products of powers of cotangent and tangent with powers of cosecant and secant respectively.

This paper, which has resulted in a cross-fertilization of the fields of classical analysis, number theory and computational/experimental mathematics, represents the introductory work of a more ambitious programme where the ideas and methods presented here

and in [18, 19] are to be extended beyond the trigonometric power sums appearing in Refs. [1] and [3]. Included in this investigation will be the cases where summands alternate in sign. Once again, the existing results give similar unwieldy results to (4.5) and thus, do not provide the interesting mathematics associated with the coefficients of the resultant polynomials.

7. ACKNOWLEDGMENT

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